I. Find f that satisfies the integral equation

$$\int_{-\infty}^{\infty} f(x-y)\, e^{-|y|}\, dy = 2e^{-|x|} - e^{-2|x|}$$

Proof: Let $g(x) = e^{-|x|}$, $h(x) = 2e^{-|x|} - e^{-2|x|}$

Then $f * g(x) = h(x)$

Recall $\widehat{f*g}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi)$.

Thus $\hat{f}(\xi)\, \hat{g}(\xi) = \hat{h}(\xi)$

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i x \xi} dx$$

$$= \int_{0}^{\infty} e^{-x} e^{-2\pi i x \xi} dx + \int_{-\infty}^{0} e^{x} e^{-2\pi i x \xi} dx$$

$$= \frac{1}{1+2\pi i \xi} + \frac{1}{1-2\pi i \xi}$$

$$= \frac{2}{1+4\pi^2 \xi^2}$$

$$\hat{h}(\xi) = \frac{4}{1+4\pi^2\xi^2} - \frac{1}{2} \frac{2}{1+4\pi^2\left(\frac{\xi}{2}\right)^2}$$

$$= \frac{4}{1+4\pi^2\xi^2} - \frac{4}{4+4\pi^2\xi^2}$$

$$= \frac{4}{1+4\pi^2\xi^2}\left(1 - \frac{1+4\pi^2\xi^2}{4+4\pi^2\xi^2}\right)$$

$$= \frac{2}{1+4\pi^2\xi^2} \, \frac{6}{4+4\pi^2\xi^2}$$

$$= \hat{g}(\xi) \frac{6}{4+4\pi^2\xi^2}$$

Therefore, $\hat{f}(\xi) = \dfrac{6}{4+4\pi^2\xi^2} = \dfrac{3}{2}\hat{k}(\xi)$

where $k(x) = e^{-2|x|}$

Put $f(x) := \frac{3}{2} e^{-2|x|}$.

Since f, g, h, $\dfrac{1}{1+4\pi^2\xi^2}$, $\dfrac{1}{4+4\pi^2\xi^2}$ are all continuous and of moderate decrease, by Fourier Inversion Formula, $f(x) = \frac{3}{2} e^{-2|x|}$ is a solution to the given equation.

II. Suppose f is continuous and of moderate decrease such that

$$\int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy} dy = 0 \quad \text{for all } x \in \mathbb{R}.$$

Then $f \equiv 0$.

Proof: Let $g(z) = e^{-z^2}$.

Then $$f * g(x) = \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2} dy$$

$$= \int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy} dy \, e^{-x^2}$$

$$= 0 \quad \text{for all } x \in \mathbb{R}$$

Then $\hat{f}(\xi) \hat{g}(\xi) = \widehat{f * g}(\xi) = 0$, $\forall \xi \in \mathbb{R}$

Recall that $e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi \xi^2}$.

Thus $\hat{g}(\xi) \neq 0$, $\forall \xi \in \mathbb{R}$.

Then $\hat{f}(\xi) = 0$, $\forall \xi \in \mathbb{R}$.

Since $f, \hat{f}$ are of moderate decrease,

by Fourier Inverse Formula, $f \equiv 0$.

III. Let $h(x) = e^{-|x|} \cos x$.

Fact: $\hat{h}(\xi) = 2 \dfrac{(2\pi\xi)^2 + 2}{(2\pi\xi)^4 + 4}$

Compute $\displaystyle\int_{-\infty}^{\infty} \left(\frac{x^2+2}{x^4+4} \right)^2 dx$

Proof: Let $g(x) = \pi\, h(2\pi x) = \pi e^{-2\pi|x|} \cos 2\pi x$

Then $\hat{g}(\xi) = \pi \cdot \dfrac{1}{2\pi} \hat{h}\left(\dfrac{\xi}{2\pi}\right) = \dfrac{\xi^2+2}{\xi^4+4}$.

By Plancherel Formula,

$$\int_{-\infty}^{\infty} \left(\frac{\xi^2+2}{\xi^4+4} \right)^2 dx = \int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 \, d\xi$$

$$= \int_{-\infty}^{\infty} |g(x)|^2 \, dx$$

$$= \pi^2 \int_{-\infty}^{\infty} e^{-4\pi|x|} (\cos 2\pi x)^2 \, dx$$

$$= 2\pi^2 \int_{0}^{\infty} e^{-4\pi x} \left(\frac{e^{2\pi i x} + e^{-2\pi i x}}{2} \right)^2 dx$$

$$= \frac{\pi^2}{2} \int_{0}^{\infty} e^{-4\pi x} \left(e^{4\pi i x} + 2 + e^{-4\pi i x} \right) dx$$

$$= \frac{\pi^2}{2} \int_0^\infty \left(2e^{-4\pi x} + e^{(-4\pi+4\pi i)x} + e^{(-4\pi-4\pi i)x} \right. dx$$

$$= \frac{\pi^2}{2} \left(- \frac{2}{-4\pi} - \frac{1}{-4\pi+4\pi i} - \frac{1}{-4\pi-4\pi i} \right)$$

$$= \frac{\pi^2}{2} \left(\frac{1}{2\pi} + \frac{1}{4\pi-4\pi i} + \frac{1}{4\pi+4\pi i} \right)$$

$$= \frac{\pi^2}{2} \left(\frac{1}{2\pi} + \frac{8\pi}{16\pi^2+16\pi^2} \right)$$

$$= \frac{\pi^2}{2} \left(\frac{1}{2\pi} + \frac{1}{4\pi} \right)$$

$$= \frac{3\pi}{8}.$$

$\square$